BOXES FOR CURVES OF CONSTANT LENGTH

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ABSTRACT

We prove that every arc of length L in E^n lies in some hypercube of diagonal L and that every closed curve of length L lies in some hypercube of diagonal $L/2$. In the case $n = 2$, we find the smallest rectangle that can accommodate every arc of length L and the smallest rectangle that can accommodate every closed curve of length L.

1. Introduction

It is known (see [7]) that each arc of length L in E^n lies in a ball of radius $L/2$, and each closed curve of length L lies in a ball of radius $L/4$. In this note we prove that the hypercubes inscribed in these balls are already large enough: each arc of length L lies in a hypercube of diagonal L, and each closed curve of length L lies in a hypercube of diagonal $L/2$.

Section 3 is devoted to the result (due to A. Meir for $n = 2$) that every arc of length L in E^n lies in some semiball of radius $L/2$. The hypercube of diagonal L is smaller than the semiball of radius $L/2$ for every $n \geqq 3$, but in the plane the semidisk is smaller than the square. In the last section we determine the rectangles of least area that can accommodate (plane) arcs and closed curves of length L. The result for closed curves depends on Cauchy's formula for the length of a closed, convex curve; and the result for arcs, found by Jones and Schaer [4], depends on the solution of the "broadworm" problem presented in [6].

2. Boxes in E^n

We begin by studying orthotopes circumscribed about a closed curve in E^n. An *orthotope* is the analog in E^n of a rectangular region in the plane and a (solid)

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rectangular parallelepiped in space. In suitable Cartesian coordinates, the orthotope T with edges $e_1, e_2, \cdots, e_n$ is the region

$$T = \{(x_1, x_2, \cdots, x_n) \colon 0 \leqq x_m \leqq e_m \text{ for } 1 \leqq m \leqq n\}. \tag{1}$$

Its main diagonal has length $d = (\sum_{m=1}^{n} e_m^2)^{\frac{1}{2}}$, and its hypervolume is $\prod_{m=1}^{n} e_m$. Its $2n$ faces (of dimension $n-1$) are parallel by pairs: for each $m = 1, 2, \cdots, n$ the faces

$$\{(x_1, x_2, \ldots, x_n) \colon x_m = e,\ 0 \leqq x_k \leqq e_k \text{ for } k \neq m\}$$

for $e = 0$ and $e = e_m$ are both normal to the x_m-axis.

We say that an orthotope T is *circumscribed* about an arc Γ if $\Gamma \subseteq T$ and if Γ meets each of the $2n$ hyperplanes that bound T. Our results depend on the following lemma, which extends to E^n a result obtained in the plane by Jones and Schaer [4].

LEMMA 1. *The diagonal of any orthotope that is circumscribed about a closed curve of length L is at most L/2.*

PROOF. Choose coordinates so that the orthotope T circumscribed about the (sensed) closed curve Γ is given by (1). Let h_1 be the hyperplane $x_1 = 0$, let $h_2, h_3, \cdots, h_{2n}$ be the remaining hyperplane faces of T indexed in an order in which they are touched by Γ, and let $P_1, P_2, \cdots, P_{2n}$ be points of Γ lying in $h_1, h_2, \cdots, h_{2n}$, respectively. Note that the hyperplanes $h_1, h_2, \cdots, h_{2n}$ are distinct, but some of the points $P_1, P_2, \cdots, P_{2n}$ may coincide. Let $P_{2n+1} = P_1$.

Write p_m^k for the x_m-coordinate of P_k, and put $s_m^k = (p_m^{k+1} - p_m^k)^2$. Plainly,

$$\sum_{k=1}^{2n} (s_m^k)^{\frac{1}{2}} = \sum_{k=1}^{2n} \left| p_m^{k+1} - p_m^k \right| \geqq 2e_m$$

for each $m = 1, 2, \cdots, n$, and

$$\left(\sum_{m=1}^{n} s_m^k \right)^{\frac{1}{2}} = P_k P_{k+1}$$

for each $k = 1, 2, \cdots, 2n$. Then according to Minkowski's inequality (see [3, 31])

$$d = \left[\sum_{m=1}^{n} e_m^2 \right]^{\frac{1}{2}} \leqq \frac{1}{2} \left[\sum_{m=1}^{n} \left(\sum_{k=1}^{2n} (s_m^k)^{\frac{1}{2}} \right)^2 \right]^{\frac{1}{2}}$$

$$\leqq \frac{1}{2} \sum_{k=1}^{2n} \left(\sum_{m=1}^{n} s_m^k \right)^{\frac{1}{2}} = \frac{1}{2} \sum_{k=1}^{2n} P_k P_{k+1} \leqq \frac{L}{2},$$

proving the assertion (cf. [4]).

Another, more geometrical proof of this Lemma can be based on reflections. For each point P and hyperplane h, denote the point symmetric to P in h by hP. If h' is a hyperplane, then $hh' = \{hP: P \in h'\}$ is also a hyperplane. If h, h', and h'' are hyperplanes, we construe $hh'h''$ from the right: $hh'h'' = \{hP: P \in h'h''\}$; and we agree that $hh'h''P$ means $h(h'(h''P))$ rather than $(hh'h'')P$. Similar conventions hold for longer strings.

Define hyperplanes $k_1, k_2,, k_{2n}$ recursively by setting $k_1 = h_1$, $k_2 = h_2$, and for $i = 2, 3, \cdots, 2n-1$,

$$k_{i+1} = k_i k_{i-1} \cdots k_2 h_{i+1},$$

and define points $Q_1, Q_2, \cdots, Q_{2n+1}$ by setting $Q_1 = P_1$, $Q_2 = P_2$, and for $i = 2, 3, \cdots, 2n$,

$$Q_{i+1} = k_i k_{i-1} \cdots k_2 P_{i+1}.$$

For each $i = 1, 2, \cdots, 2n$, the point Q_i lies on the hyperplane k_i because P_i lies on the hyperplane h_i, and so

$$Q_i = k_i Q_i = k_i k_{i-1} \cdots k_2 P_i.$$

Successive reflections through the $i-1$ hyperplanes $k_2, k_3, \cdots, k_i$ carry both P_i to Q_i and P_{i+1} to Q_{i+1}. Consequently $P_i P_{i+1} = Q_i Q_{i+1}$ for each i, and

$$\sum_{i=1}^{2n} P_i P_{i+1} = \sum_{i=1}^{2n} Q_i Q_{i+1}.$$

The point Q_{2n+1} is obtained by reflecting P_1 successively through the hyperplanes $k_1, k_2, \cdots, k_{2n+1}, k_{2n}$, in that order. Any two of these $2n$ hyperplanes are either parallel or perpendicular, and each is parallel to exactly one other. Since reflections in perpendicular hyperplanes commute, we can rearrange the sequence of reflections so that the parallel hyperplanes are adjacent:

$$Q_{2n+1} = k_{2n} k_{2n-1} \cdots k_2 k_1 P_1 = k'_{2n} k'_{2n-1} \cdots k'_4 k'_3 k'_2 k'_1 P_1$$

where for each $i = 1, 2, \cdots, n$, the hyperplanes k'_{2i-1} and $2'_{2i}$ are parallel. Since the product of reflections in two parallel hyperplanes is a translation through twice the distance between them, it is evident that if Q_1 $(= P_1)$ has coordinates $(p_1, p_2, \cdots, p_n)$ and Q_{2n+1} has coordinates $(q_1, q_2, \cdots, q_n)$, then $|q_m - p_m| = 2e_m$ for each $m = 1, 2, \cdots, n$. Thus

$$Q_1 Q_{2n+1} = \left(\sum_{m=1}^{n} |q_m - p_m|^2 \right)^{\frac{1}{2}} = 2\left(\sum_{m=1}^{n} e_m^2 \right)^{\frac{1}{2}} = 2d,$$

where d is the diagonal of T. Consequently,

$$L \geqq \sum_{i=1}^{2n} P_iP_{i+1} = \sum_{i=1}^{2n} Q_iQ_{i+1} \geqq Q_1Q_{2n+1} = 2d.$$

and so $d \leqq L/2$.

The equality $d = L/2$ holds precisely for those closed polygonal curves $\Gamma = [P_1P_2 \cdots P_{2n}P_1]$ that "unfold" to the segment $[Q_1Q_{2n+1}]$, including the diagonal segment of the orthotope (traversed twice).

The assumption that $\Gamma \subseteq T$ was never needed in the proof, so we have actually established a little more:

COROLLARY 2. *Every closed curve that meets each of the* $2n$ *hyperplanes that bound an orthotope of diagonal* d *in* E^n *has length at least* $2d$; *and every arc that meets each of these hyperplanes has length at least* d.

According to a well-known theorem of Kakutani [5] (for $n = 3$) and to Yamabe and Yujobô [9] (for arbitrary n), any compact, convex set in E^n has a circumscribed hypercube. Let G be the closed, convex hull of a closed curve Γ of length L, and let T be a hypercube circumscribed about G. Suppose that T has edge e and diagonal d. Then since every supporting hyperplane of G contains a point of Γ, the hypercube T is circumscribed about Γ; and according to the Lemma, $e = d/\sqrt{n} \leqq L/(2\sqrt{n})$. Thus,

THEOREM 3. *Every closed curve of length* L *in* E^n *lies in some hypercube of edge* $L/(2\sqrt{n})$, *but no smaller hypercube contains a congruent copy of every closed curve of length* L.

Now suppose that Γ is an arc of length L, and let Γ' be the closed curve of length $L' \leqq 2L$ that is formed when the endpoints of Γ are connected by a segment. According to the Theorem, Γ' lies in some hypercube od edge $L'/(2\sqrt{n}) \leqq L/\sqrt{n}$.

THEOREM 4. *Every arc of length* L *in* E^n *lies in some hypercube of edge* $L/\sqrt{n}$, *but no smaller hypercube contains a congruent copy of every arc of length* L.

3. Semiballs in E^n

In answer to a question raised by Leo Moser, A. Meir proved some years ago that every plane arc of unit length lies in some closed semidisk of radius $\frac{1}{2}$. His elegant argument applies virtually unchanged in E^n. A semiball is a region of the form $B \cap H$, where B is a (solid) ball and H is a closed halfspace whose face is a hyperplane through the center of B.

THEOREM 5. *Every arc of length L in E^n lies in some semiball of radius $L/2$, but no smaller semiball contains a congruent copy of every arc of length L.*

PROOF. If Γ is a closed arc of length L, let h be any supporting hyperplane of Γ, let H be the closed halfspace with face h that contains Γ, let R be any point of contact of Γ with h, and let B be the ball of radius $L/2$ centered at R. Then Γ obviously lies in the semiball $B \cap H$.

If Γ has distinct endpoints P and Q, let h be any supporting hyperplane of Γ that is parallel to the line PQ, let H be the closed halfspace with face h that contains Γ, let R be any point of contact of Γ with h, and let P' and Q' be the points symmetric to P and Q in h. The segments $[PQ']$ and $[QP']$ meet h at a point O. If X is any point of Γ that lies between P and R in Γ, then (cf. [7])

$$OX \leqq \tfrac{1}{2}(XP + XQ') \leqq \tfrac{1}{2}(PX + XR + RQ) \leqq \frac{L}{2}.$$

Similarly, $OX \leqq L/2$ if X lies between R and Q on Γ. Thus Γ lies in the semiball $B \cap H$, where B is the ball of radius $L/2$ centered at O.

The hypervolume S_n of a semiball of radius $L/2$ in E^n is given by (see [2, 125 f.])

$$S_n = \begin{cases} \dfrac{\pi^m L^n}{2^{n+1} m!} & \text{when } n = 2m \\[2ex] \dfrac{\pi^m m! L^n}{2n!} & \text{when } n = 2m + 1 \end{cases}$$

and the hypervolume C_n of a hypercube of diagonal L in E^n is $C_n = L^n/n^{n/2}$. One can verify that $C_n < S_n$ for every $n \geqq 3$, but $S_2 < C_2$.

4. Rectangles in E^2

In this section we determine the rectangles of least area that can accommodate every plane closed curve of length L and every plane arc of length L. It is apparently not known whether the rectangles described here are the best (i.e., smallest) convex quadrilaterals, nor are analogous results known for $n \geqq 3$.

The argument for closed curves is simplified by the observation that it is enough to consider closed, convex curves. The length L' of the closed, convex curve that bounds the convex hull of a closed curve Γ of length L is at most L (a proof of this well-known fact is given in [6]), and so, by a suitable dilatation, one can produce a convex curve of length L that surrounds Γ. Consequently, a convex region can accommodate all closed curves of length L if and only if it can accommodate all closed, convex curves of length L.

THEOREM 6. *Among all rectangles that can accommodate all closed curves of length L, the rectangle with least area has sides L/π and $L(\pi^2-4)^{\frac{1}{2}}/(2\pi)$ and area approximately* $0.122737L^2$.

PROOF. Without loss of generality we assume that $L=1$. According to Cauchy's formula (see [1, 48f.]), if $w(\theta)$ is the width in the direction θ of a closed, convex curve Γ of length 1, then

$$\frac{1}{2\pi}\int_0^{2\pi} w(\theta)d\theta = \frac{1}{\pi}.$$

Consequently there exist directions θ_1 and θ_2 such that

$$w(\theta_1) \leqq \frac{1}{\pi} \leqq w(\theta_2),$$

and by the continuity of $w(\theta)$ there is a direction θ_0 such that $w(\theta_0) = \pi^{-1}$. The rectangle circumscribed about Γ with one side in the direction θ_0 has a side of length π^{-1}; let a be the length of the other side. By Lemma 1, $a \leqq (2^{-2}-\pi^{-2})^{\frac{1}{2}}$. Thus the rectangle with sides π^{-1} and $(\pi^2-4)^{\frac{1}{2}}/(2\pi)$ can accommodate Γ.

No smaller rectangle can accommodate every such closed curve, because one side must be at least π^{-1} (so that a circle of circumference 1 can be accommodated) and the diagonal must be at least $\frac{1}{2}$ (so that a segment of length $\frac{1}{2}$ can be accommodated).

It is worth remarking that the area of any closed, convex set that contains a congruent copy of every closed curve of length L must be at least

$$\frac{L^2}{4\pi^2}\left[(\pi^2 - 4)^{\frac{1}{2}} + \pi - 2\arccos\frac{2}{\pi}\right] \approx 0.096330L^2,$$

for this is the area of the smallest convex set spanned by a segment of length $L/2$ and a circle of circumference L. It is easy to see that the minimal configuration is the arrangement in which the midpoint of the segment is the center of the circle.

The corresponding result for arcs depends on the solution of the "broadworm" problem, obtained in [6]:

LEMMA 7. *Let*

$$\alpha = \arcsin\left[\frac{1}{6} + \frac{4}{3}\sin\left(\frac{1}{3}\arcsin\frac{17}{64}\right)\right] \approx 0.290046$$

$$\gamma = \arctan\left(\frac{1}{2}\sec\alpha\right) \approx 0.480931$$

$$\beta = \frac{\pi}{2} - \alpha - 2\gamma \approx 0.318888$$

$$b_0 = \frac{1}{2}(\beta + \tan\alpha + \tan\gamma)^{-1} \approx 0.438925.$$

Then every arc of length 1 *lies in some infinite strip of width* b_0, *and there is a unique arc of length* 1 (*called the broadworm*, see Fig. 1) *that lies in no narrower strip.*

Using this result, we can determine the smallest rectangle that can accommodate every arc of length L, a result found by Jones and Schaer [4].

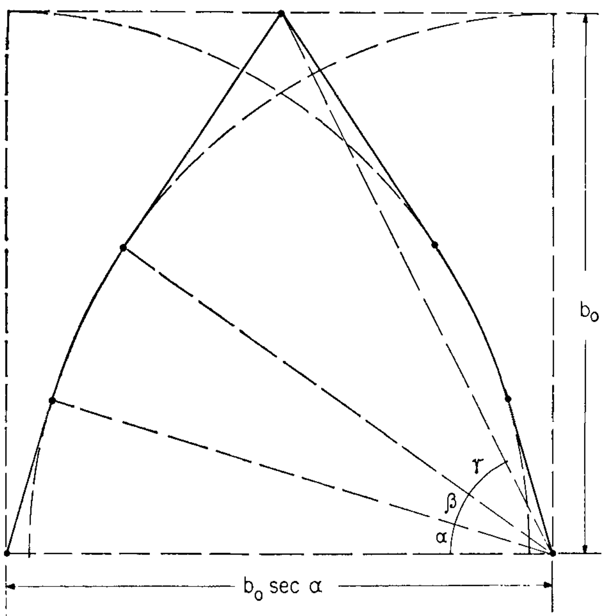

Fig. 1. The broadworm

THEOREM 8. *Among all rectangles that can accommodate all arcs of length* L, *the rectangle with least area has sides* Lb_0 *and* $L(1-b_0^2)^{\frac{1}{2}}$ *and area approximately* 0.394385 L^2.

PROOF. Without loss of generality we assume that $L = 1$. Let Γ be an arc of length 1, and for each direction θ let $w(\theta)$ be the width of Γ in the direction θ. If $w(\theta) < b_0$ for every θ, then Γ surely lies in the rectangle with sides b_0 and $(1 - b_0^2)^{\frac{1}{2}}$, because $(1 - b_0^2)^{\frac{1}{2}} > b_0$. Otherwise by continuity there is a direction θ_0 in which Γ has width $w(\theta_0) = b_0$. The rectangle circumscribed about Γ with one side in the direction θ_0 has a side of length b_0; let a be the length of the other side. Joining the endpoints of Γ by a segment gives a closed curve of length at most 2 about which the rectangle is circumscribed, and according to Lemma 1, $a \leqq (1 - b_0^2)^{\frac{1}{2}}$. Thus the rectangle with sides of length b_0 and $(1 - b_0^2)^{\frac{1}{2}}$ can accommodate every unit arc.

No smaller rectangle can accommodate every such arc, because one side must be at least b_0 (so that the broadworm can be accommodated) and the diagonal must be at least 1 (so that the unit segment can be accommodated).

The area of any closed, convex set that contains a congruent copy of every arc of length L must be at least $b_0L^2/2 \approx 0.219463\,L^2$. This follows from the fact that the diameter of such a set must be at least L and its width in the direction perpendicular to that diameter must be at least b_0L. Meir's semidisk, with area $\pi L^2/8 \approx 0.392699\,L^2$, is a little smaller than the smallest rectangle. It is known (see [8]) that a certain truncated sector of area less than $0.3443\,L^2$ contains a congruent copy of every arc of length L, but the gap between this area and the lower bound given above is quite wide.

REMARK. Since this paper was submitted for publication, it has come to our attention that many of these results have been found independently by G. D. Chakerian and M. S. Klamkin (*Minimal Covers for Closed Curves*, to appear in Mathematics Magazine).

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